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Free Vibration and Buckling of Symmetric Cross-Ply Rectangular Laminates

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The method of superposition is further developed to analyze free vibration and buckling of symmetric cross-ply laminated rectangular plates. The improved plate theory developed by Mindlin is implemented. A four building block technique is employed that reduces the number of combinations of classical boundary conditions that need to be studied. Combinations of clamped and simply supported boundary conditions are investigated. Eigenvalues and critical buckling loads with four-digit accuracy are obtained for plates with five distinct combinations of simply supported and clamped boundary conditions.

Nomenclature

A_{44}, A_{55}	= extensional rigidity parameters
a_{44}, a_{55}	= extensional rigidity parameters divided by $E_2 h$
a	= plate dimension in the x direction
b	= plate dimension in the y direction
C	= clamp
$D_{11}, D_{22}, D_{12}, D_{66}$	= flexural rigidity parameters
$d_{11}, d_{22}, d_{12}, d_{66}$	= flexural rigidity parameters divided by $E_2 h^3$
E_1, E_2	= Young's moduli of individual ply in the direction parallel and perpendicular to fibers, respectively
G_{12}, G_{23}, G_{31}	= shear moduli in three coordinate planes of unidirectional composite
h_p	= transverse distance from the upper surface of pth ply to the midplane
$\mathcal{M}, \mathcal{N}$	= fabrication parameters
P_x, P_y	= in-plane forces in the x and y directions, respectively
P_ξ	= nondimensional in-plane force, $P_x b^2/E_2 h^3$
P_η	= nondimensional in-plane force, $P_y a^2/E_2 h^3$
P_ξ^{cr}	= critical buckling load in the x direction
S	= simple support
s	= length-to-thickness ratio, a/h
x, y, z	= Cartesian coordinates
α	= load ratio, P_y/P_x
η	= nondimensional in-plane coordinates, y/b
η_c	= plate central axis parallel to η axis
κ^2	= shear correction factor, 0.8601
λ^2	= eigenvalue of free vibration or frequency parameter, $\omega a^2\sqrt{\rho/E_2 h^3}$
ν_{12}	= major Poisson's ratio of unidirectional composite
ν_{21}	= minor Poisson's ratio of unidirectional composite
ξ	= nondimensional in-plane coordinate, x/a
ξ_c	= plate central axis parallel to ξ axis
ρ	= density per unit area
ϕ	= plate aspect ratio, a/b
ω	= natural frequency of plate free vibration

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Introduction

AS an important structural element, fiber-reinforced laminated plates with anisotropic mechanical properties find wide applications in various fields of engineering. Accurate knowledge of their dynamic behavior is essential in the design of plates made of composite materials.

Different plate theories have been developed in analyzing laminated plates. The classical plate theory (CPT) based on the Kirchhoff hypothesis overpredicts natural frequencies and critical uckling loads. The degree of this overprediction can be very significant, especially for those plates having low length-to-thickness ratio and high in-plane elasticity ratio. Refined theories based on discarding the Kirchhoff hypothesis and incorporating transverse shear deformation were subsequently developed. Among those refined theories, the improved plate theory (IPT) developed by Mindlin[1] is simple and adequate for free vibration and buckling analysis of moderately thick laminated plates. A comparison prepared in this paper indicates that there is an excellent agreement between results using IPT and those of higher order plate theories. Therefore, the IPT is adopted in this paper.

Because the governing equations for free vibration and buckling are coupled partial differential equations, the importance of boundary conditions in solving these equations cannot be overlooked. There have been no exact solutions in the literature for rectangular laminates with arbitrary classical boundary conditions, except for those having at least one pair of opposite edges simply supported. The Navier method[2,3] is used to find analytical solutions for the simply supported composite plates. Solutions for plates with one pair of opposite edges simply supported may be obtained using the Lévy method.[4,5]

Reviewing two comprehensive survey articles by Leissa,[6,7] one finds that little work has been done in accurate analysis of free vibration and buckling of composite plates with arbitrary classical boundary conditions. Most results published in the literature are approximate in nature. Among the approximate analytical approaches, the Rayleigh-Ritz energy method is most commonly used in free vibration and buckling analysis of composite plates such as in the work of Reddy,[8] Dickinson,[9] and Roufaeil and Dawe.[10] Numerical methods, such as the finite element method,[11] are also used in the literature. Compared with the analytical methods, numerical methods usually require more computations.

The method of superposition introduced by Gorman[12] in analyzing free vibration of thin isotropic plates has been used to handle the accurate analysis of free vibration[13] and buckling[14] of thin orthotropic rectangular plates using the classical plate theory. In this paper, the method of superposition is further developed to analyze free vibration and buckling of symmetrically laminated cross-ply rectangular plates with combinations of clamped and simply supported edges using the IPT and the four building block technique. Since four building blocks are used, solutions for plates with up to three edges simply supported are obtainable from that for the fully clamped plate by simply eliminating contributions of building blocks associated with the simply supported edges.

Among the advantages of the method of superposition and the four building block technique is the significant reduction in the number of boundary conditions that have to be investigated toward completion of the free vibration and buckling analysis of rectangular laminates with arbitrary combinations of classical boundary conditions, i.e., simple support, clamp, free, and slip shear. Using the procedure developed in this paper, one only needs to investigate laminates with the following six combinations of boundary conditions: clamped along all four edges (dealt with in this paper), clamped along three edges and free along the other edge, clamped along two adjacent edges and free along the other two adjacent edges, clamped along two opposite edges and free along the other two opposite edges, clamped along one edge and free along the other three edges, and free along all four edges. It is interesting to notice that two boundary conditions, simple support and slip shear, are not among the six combinations because solutions for plates having edges subjected to these two boundary conditions may be obtained from the solutions for the studied cases.

Using the four building block technique, symmetry and antisymmetry of vibration modes with respect to the two central axes within the midplane of a plate may also be identified by simply restructuring the general algebraic equations. For a cross-ply laminate, such a symmetry or antisymmetry about its central axes is evidenced by the identity of boundary conditions imposed on the two pairs of opposite edges. For example, boundary conditions for a fully clamped plate are identical for any pair of opposite edges. Therefore, all modes of the fully clamped plate must be either symmetric or antisymmetric about the two central axes. Consequently, a fully clamped symmetric cross-ply laminated rectangular plate has four families of vibration modes.

Numerical calculations indicate that the method of superposition converges quickly for different vibration modes, boundary conditions, and material properties. Since solutions obtained by the method satisfy the governing differential equations exactly and the boundary conditions to an arbitrary degree of accuracy, results presented in this paper are not only convergent but also accurate.

Governing Differential Equations

The governing differential equations for free vibration of symmetric cross-ply laminated plates subjected to in-plane loads P_x and P_y may be written as

$$\kappa^2 A_{55}\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+\kappa^2 A_{44}\left(\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)$$
$$-P_x\frac{\partial^2 w}{\partial x^2}-P_y\frac{\partial^2 w}{\partial y^2}+\rho\omega^2 w = 0 \tag{1}$$

$$D_{11}\frac{\partial^2\psi_x}{\partial x^2}+D_{66}\frac{\partial^2\psi_x}{\partial y^2}+(D_{12}+D_{66})\frac{\partial^2\psi_y}{\partial x\partial y}$$
$$-\kappa^2 A_{55}\left(\psi_x+\frac{\partial w}{\partial x}\right)+\frac{1}{12}\rho h^2\omega^2\psi_x = 0 \tag{2}$$

$$D_{22}\frac{\partial^2\psi_y}{\partial y^2}+D_{66}\frac{\partial^2\psi_y}{\partial x^2}+(D_{12}+D_{66})\frac{\partial^2\psi_x}{\partial x\partial y}$$
$$-\kappa^2 A_{44}\left(\psi_y+\frac{\partial w}{\partial y}\right)+\frac{1}{12}\rho h^2\omega^2\psi_y = 0 \tag{3}$$

In the preceding equations, $w(x, y)$ is the lateral displacement; $\psi_x(x, y)$ and $\psi_y(x, y)$ are the angles between the deformed and original material lines normal to the midplane, measured in the coordinate planes xoz and yoz, respectively.

To describe the boundary conditions in terms of displacements, the following expressions relating moments and shear forces to the displacements are useful:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{Bmatrix} D_{11}\frac{\partial\psi_x}{\partial x}+D_{12}\frac{\partial\psi_y}{\partial y} \\ D_{22}\frac{\partial\psi_y}{\partial y}+D_{12}\frac{\partial\psi_x}{\partial x} \\ D_{66}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right) \\ \kappa^2 A_{55}\left(\psi_x+\frac{\partial w}{\partial x}\right) \\ \kappa^2 A_{44}\left(\psi_y+\frac{\partial w}{\partial y}\right) \end{Bmatrix} \tag{4}$$

Experience has shown that it is advantageous to work with nondimensional quantities in free vibration and buckling analysis. Therefore, the following transformations are introduced:

$$x = a\xi, \quad y = b\eta, \quad w = aW, \quad \psi_x = \Psi_\xi, \quad \psi_y = \Psi_\eta \tag{5}$$

The governing differential equations may now be rewritten as

$$\kappa^2 a_{55}\left(\frac{\partial\Psi_\xi}{\partial\xi}+\frac{\partial^2 W}{\partial\xi^2}\right)+\kappa^2\phi a_{44}\left(\frac{\partial\Psi_\eta}{\partial\eta}+\phi\frac{\partial^2 W}{\partial\eta^2}\right)$$
$$-\frac{\phi^2}{s^2}\left(P_\xi\frac{\partial^2 W}{\partial\xi^2}+P_\eta\frac{\partial^2 W}{\partial\eta^2}\right)+\frac{\lambda^4}{s^2}W = 0 \tag{6}$$

$$d_{11}\frac{\partial^2\Psi_\xi}{\partial\xi^2}+d_{66}\phi^2\frac{\partial^2\Psi_\xi}{\partial\eta^2}+(d_{12}+d_{66})\phi\frac{\partial^2\Psi_\eta}{\partial\xi\partial\eta}$$
$$-\kappa^2 s^2 a_{55}\left(\Psi_\xi+\frac{\partial W}{\partial\xi}\right)+\frac{\lambda^4}{12s^2}\Psi_\xi = 0 \tag{7}$$

$$d_{22}\phi^2\frac{\partial^2\Psi_\eta}{\partial\eta^2}+d_{66}\frac{\partial^2\Psi_\eta}{\partial\xi^2}+(d_{12}+d_{66})\phi\frac{\partial^2\Psi_\xi}{\partial\xi\partial\eta}$$
$$-\kappa^2 s^2 a_{44}\left(\Psi_\eta+\phi\frac{\partial W}{\partial\eta}\right)+\frac{\lambda^4}{12s^2}\Psi_\eta = 0 \tag{8}$$

where the frequency parameter or eigenvalue of free vibration λ^2, the plate aspect ratio ϕ, the length-to-thickness ratio s, and other nondimensional parameters appearing in the preceding equations are defined in the Nomenclature. In buckling analysis, a load ratio $\alpha = P_y/P_x$ relating the two in-plane loads is introduced. The nondimensional loads P_ξ and P_η are related by $P_\eta = P_\xi\alpha\phi^2$.

Differential equations (6–8) are solved by letting $P_\xi = 0$ and $P_\eta = 0$ for free vibration analysis and by letting $\lambda^2 = 0$ for buckling analysis. Effects of in-plane loads on free vibration of laminates will not be investigated in this paper.

The nondimensional moments and shear forces may be written in terms of nondimensional quantities as

$$\begin{Bmatrix} M_\xi \\ M_\eta \\ M_{\xi\eta} \\ Q_\xi \\ Q_\eta \end{Bmatrix} = \begin{Bmatrix} \frac{\partial\Psi_\xi}{\partial\xi}+\frac{\phi d_{12}}{d_{11}}\frac{\partial\Psi_\eta}{\partial\eta} \\ \frac{\partial\Psi_\eta}{\partial\eta}+\frac{d_{12}}{\phi d_{22}}\frac{\partial\Psi_\xi}{\partial\xi} \\ \phi\frac{\partial\Psi_\xi}{\partial\eta}+\frac{\partial\Psi_\eta}{\partial\xi} \\ \Psi_\xi+\frac{\partial W}{\partial\xi} \\ \Psi_\eta+\phi\frac{\partial W}{\partial\eta} \end{Bmatrix} \tag{9}$$

Analytical Solutions

To obtain accurate analytical solutions to the partial differential equations, we use four building blocks shown in Fig. 1, which meet the following requirements:

1) Each building block has the dimensions and loadings identical to the original laminate.

2) There exists a pair of simply supported opposite edges in each building block.

3) Two of the three boundary conditions for the original laminate are satisfied by each building block.

4) A distributed bending moment is imposed on different edges consecutively to fulfill the boundary condition requirements, where the bending moment itself contains k unknown coefficients that determine the mode shapes of free vibration and buckling.

The building blocks play an important role in the accurate analysis of free vibration and buckling of a rectangular laminate because an exact solution for each building block may be obtained using the Lévy method. Since the governing differential equations are linear, solutions for the four building blocks may be superimposed to form a solution for the original laminate. For the superimposed solution to be a true solution for the original plate, the third boundary condition released along the four rectilinear edges of the original plate during the construction of building blocks must be enforced. From these conditions, one may obtain eigenvalues of free vibration and buckling and their corresponding mode shapes.

For convenience, a summary of Lévy type solutions and expressions for boundary conditions for the four building blocks is provided in Tables 1 and 2, respectively. In Table 2, an overdot represents the first derivative with respect to ξ for the second and fourth building blocks and with respect to η for the first and third building blocks.

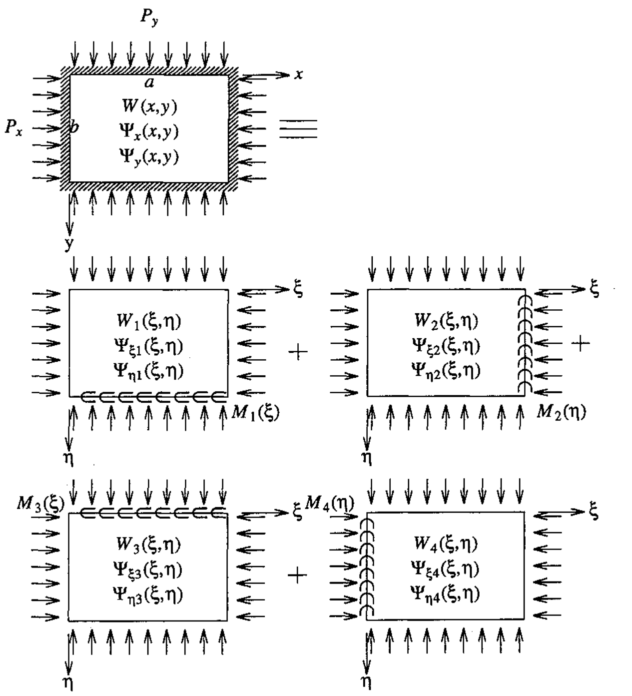

Fig. 1 Building blocks used in vibration and buckling analysis of rectangular laminates.

Solution for the First Building Block

Substituting the Lévy type solution for the first building block into Eqs. (6–8), one obtains the following ordinary differential equations:

$$\begin{Bmatrix} \ddot{X}_m^{(1)} \\ \ddot{Y}_m^{(1)} \\ \ddot{Z}_m^{(1)} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & a_{m1} \\ 0 & 0 & a_{m2} \\ a_{m3} & a_{m4} & 0 \end{bmatrix} \begin{Bmatrix} \dot{X}_m^{(1)} \\ \dot{Y}_m^{(1)} \\ \dot{Z}_m^{(1)} \end{Bmatrix} + \left(\begin{bmatrix} b_{m1} & b_{m2} & 0 \\ b_{m3} & b_{m4} & 0 \\ 0 & 0 & b_{m5} \end{bmatrix} \begin{Bmatrix} X_m^{(1)} \\ Y_m^{(1)} \\ Z_m^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \right) \tag{10}$$

where $X_m^{(1)}(\eta), Y_m^{(1)}(\eta)$, and $Z_m^{(1)}(\eta), m = 1, 2, \ldots, k$, are unknown functions of η; k is the number of terms used in the Lévy type solution; single and double dots represent the first and second derivatives with respect to η, respectively; and the coefficients appearing in the preceding equations are

$$\begin{pmatrix} a_{m1} \\ a_{m2} \\ a_{m3} \\ a_{m4} \end{pmatrix} = \begin{bmatrix} \dfrac{\kappa^2 a_{44}}{(\kappa^2 a_{44} - P_\eta/s^2)\,\phi} \\ \dfrac{(d_{12} + d_{66})\, m\pi}{d_{66}\phi} \\ -\dfrac{\kappa^2 s^2 a_{44}}{d_{22}\phi} \\ -\dfrac{(d_{12} + d_{66})\, m\pi}{d_{22}\phi} \end{bmatrix}$$

$$\begin{bmatrix} b_{m1} \\ b_{m2} \\ b_{m3} \\ b_{m4} \\ b_{m5} \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda^4/s^2 + (\phi^2 P_\xi/s^2 - \kappa^2 a_{55})\,(m\pi)^2}{(\kappa^2 a_{44} - P_\eta/s^2)\,\phi^2} \\ -\dfrac{\kappa^2 a_{55} m\pi}{(\kappa^2 a_{44} - P_\eta/s^2)\,\phi^2} \\ -\dfrac{\kappa^2 s^2 a_{55} m\pi}{d_{66}\phi_2} \\ \dfrac{1}{d_{66}\phi^2}\left[\dfrac{\lambda^4}{12 s^2} - d_{11}\,(m\pi)^2 - \kappa^2 s^2 a_{55}\right] \\ \dfrac{1}{d_{22}\phi^2}\left[\dfrac{\lambda^4}{12 s^2} - d_{66}\,(m\pi)^2 - \kappa^2 s^2 a_{44}\right] \end{bmatrix}$$

Substituting into Eq. (10) the following solution

$$\begin{Bmatrix} X_m^{(1)}\eta \\ Y_m^{(1)}\eta \\ Z_m^{(1)}\eta \end{Bmatrix} = \begin{Bmatrix} A_m^* \\ B_m^* \\ C_m^* \end{Bmatrix} e^{\varepsilon_m \eta} \tag{11}$$

Table 1 Lévy solutions of the four building blocks

Building blocks	Plate displacements and nondimensional bending moments			
	W_i	$\Psi_{\xi i}$	$\Psi_{\eta i}$	M_i
1	$\Sigma_1^k X_m^{(1)}(\eta)\sin m\pi\xi$	$\Sigma_1^k Y_m^{(1)}(\eta)\cos m\pi\xi$	$\Sigma_1^k Z_m^{(1)}(\eta)\sin m\pi\xi$	$\Sigma_1^k \Gamma_m^{(1)}\sin m\pi\xi$
2	$\Sigma_1^k X_n^{(2)}(\xi)\sin n\pi\eta$	$\Sigma_1^k Z_n^{(2)}(\xi)\sin n\pi\eta$	$\Sigma_1^k Y_n^{(2)}(\xi)\cos n\pi\eta$	$\Sigma_1^k \Gamma_n^{(2)}\sin n\pi\eta$
3	$\Sigma_1^k X_m^{(3)}(\eta)\sin m\pi\xi$	$\Sigma_1^k Y_m^{(3)}(\eta)\cos m\pi\xi$	$\Sigma_1^k Z_m^{(3)}(\eta)\sin m\pi\xi$	$\Sigma_1^k \Gamma_m^{(3)}\sin m\pi\xi$
4	$\Sigma_1^k X_n^{(4)}(\xi)\sin n\pi\eta$	$\Sigma_1^k Z_n^{(4)}(\xi)\sin n\pi\eta$	$\Sigma_1^k Y_n^{(4)}(\xi)\cos n\pi\eta$	$\Sigma_1^k \Gamma_n^{(4)}\sin n\pi\eta$

Table 2 Boundary conditions for the building blocks

Edges	Building blocks 1	2	3	4
$\xi = 0$	$X_m^{(1)} = 0$	$X_n^{(2)} = 0$	$X_m^{(3)} = 0$	$X_n^{(4)} = 0$
or	$Y_m^{(1)} = 0$	$Y_n^{(2)} = 0$	$Y_m^{(3)} = 0$	$Y_n^{(4)} = 0$
$\eta = 0$	$\dot{Z}_m^{(1)} = 0$	$\dot{Z}_n^{(2)} = 0$	$\dot{Z}_m^{(3)} = \Gamma_m^{(3)}$	$\dot{Z}_n^{(4)} = \Gamma_n^{(4)}$
$\xi = 1$	$X_m^{(1)} = 0$	$X_n^{(2)} = 0$	$X_m^{(3)} = 0$	$X_n^{(4)} = 0$
or	$Y_m^{(1)} = 0$	$Y_n^{(2)} = 0$	$Y_m^{(3)} = 0$	$Y_n^{(4)} = 0$
$\eta = 1$	$\dot{Z}_m^{(1)} = \Gamma_m^{(1)}$	$\dot{Z}_n^{(2)} = \Gamma_n^{(2)}$	$\dot{Z}_m^{(3)} = 0$	$\dot{Z}_n^{(4)} = 0$

one obtains

$$\begin{bmatrix} \varepsilon_m^2 + b_{m1} & b_{m2} & a_{m1}\varepsilon_m \\ b_{m3} & \varepsilon_m^2 + b_{m4} & a_{m2}\varepsilon_m \\ a_{m3}\varepsilon_m & a_{m4}\varepsilon_m & \varepsilon_m^2 + b_{m5} \end{bmatrix} \begin{Bmatrix} A_m^* \\ B_m^* \\ C_m^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{12}$$

where ε_m are characteristic roots to be determined, and A_m^*, B_m^*, and C_m^* are real or complex constants depending on whether ε_m are real or complex. The condition for a nontrivial solution in the form of Eq. (11) requires that the determinant of the coefficient matrix in Eq. (12) vanishes. Therefore, we have

$$\varepsilon_m^6 + p_{m2}\varepsilon_m^4 + p_{m1}\varepsilon_m^2 + p_{m0} = 0 \tag{13}$$

where p_{m0}, p_{m1}, and p_{m2} are known constants.

Equation (13) is a cubic algebraic equation of ε^2. Its discriminant determining the type of analytical solutions is defined as

$$\Delta_m = r_m^2 + q_m^3 \tag{14}$$

where

$$q_m = \frac{1}{3}p_{m1} - \frac{1}{9}p_{m2}^2, \qquad r_m = \frac{1}{2}p_{m0} + \frac{1}{27}p_{m2}^3 - \frac{1}{6}p_{m1}p_{m2}$$

A cubic equation with real coefficients may have 1) three distinct real roots if Δ_m is negative, 2) a single real root and two conjugate complex roots if Δ_m is positive, and 3) three real roots of which at least two of them are identical if Δ_m is equal to zero. However, in running the computer code prepared for the analysis, we found that a zero of the discriminant Δ_m never occurred. Therefore, only the first two cases need to be discussed.

For convenience in presenting analytical solutions, the following notations are introduced:

$$\widetilde{\sinh}\,\varepsilon x = \frac{\sinh \varepsilon x}{\sinh \varepsilon} \qquad \widetilde{\sin}\,\varepsilon x = \frac{\sin \varepsilon x}{\sin \varepsilon} \qquad \widetilde{\mathrm{sn}}\,\varepsilon x = \frac{\mathrm{sn}\,\varepsilon x}{\mathrm{sn}\,\varepsilon}$$

$$\widetilde{\cosh}\,\varepsilon x = \frac{\cosh \varepsilon x}{\cosh \varepsilon} \qquad \widetilde{\cos}\,\varepsilon x = \frac{\cos \varepsilon x}{\cos \varepsilon} \qquad \widetilde{\mathrm{cs}}\,\varepsilon x = \frac{\mathrm{cs}\,\varepsilon x}{\mathrm{cs}\,\varepsilon}$$

where ε is a real, positive quantity, and x is a real variable in the interval [0, 1].

Values of functions $\widetilde{\sinh}$ and $\widetilde{\cosh}$ are now limited to be within the range from 0 to 1. Therefore, since plates with four simply supported edges are solved using the Navier method, the possibility $\varepsilon = m\pi$ that may cause singularities in the newly introduced functions $\widetilde{\mathrm{sn}}$ and $\widetilde{\sin}$ is excluded. Introductions of these notations are also found to be necessary in solving overflow problems when running the computer code.

Case $\Delta_m < 0$

In this case, the six characteristic roots of Eq. (13) are given by

$$\varepsilon_{m1,\,m2} = \pm\alpha_m \text{ or } \pm i\alpha_m, \qquad \varepsilon_{m3,\,m4} = \pm\beta_m \text{ or } \pm i\beta_m$$
$$\varepsilon_{m5,\,m6} = \pm\gamma_m \text{ or } \pm i\gamma_m \tag{15}$$

where α_m, β_m, and γ_m are known real positive numbers. The general solution for $X_m^{(1)}$, $Y_m^{(1)}$, and $Z_m^{(1)}$ may be written as

$$X_m^{(1)} = A_m \mathrm{cs}\ \alpha_m\eta + B_m \mathrm{sn}\ \alpha_m\eta + C_m \mathrm{cs}\ \beta_m\eta + D_m \mathrm{sn}\ \beta_m\eta + E_m \mathrm{cs}\ \gamma_m\eta + F_m \mathrm{sn}\ \gamma_m\eta \tag{16}$$

$$Y_m^{(1)} = R_{m1}(A_m \mathrm{cs}\ \alpha_m\eta + B_m \mathrm{sn}\ \alpha_m\eta) + R_{m2}(C_m \mathrm{cs}\ \beta_m\eta + D_m \mathrm{sn}\ \beta_m\eta) + R_{m3}(E_m \mathrm{cs}\ \gamma_m\eta + F_m \mathrm{sn}\ \gamma_m\eta) \tag{17}$$

$$Z_m^{(1)} = S_{m1}(B_m \mathrm{cs}\ \alpha_m\eta \pm A_m \mathrm{sn}\ \alpha_m\eta) + R_{m2}(D_m \mathrm{cs}\ \beta_m\eta \pm C_m \mathrm{sn}\ \beta_m\eta) + S_{m3}(F_m \mathrm{cs}\ \gamma_m\eta \pm E_m \mathrm{sn}\ \gamma_m\eta) \tag{18}$$

where R_{mj} and S_{mj}, $j = 1, 2, 3$, are real constants; $\mathrm{cs}(\alpha_m\eta)$ is a cosine function or a hyperbolic cosine function depending on whether $i\alpha_m$ or α_m is the characteristic root; $\mathrm{sn}(\alpha_m\eta)$ is a sine function or a hyperbolic sine function depending on whether $i\alpha_m$ or α_m is the characteristic root.

From the boundary conditions on $\eta = 0$ and 1, one may find the unknown constants in Eqs. (16–18) in terms of $\Gamma_m^{(1)}$. The unknown functions $X_m^{(1)}(\eta)$, $Y_m^{(1)}(\eta)$, and $Z_m^{(1)}(\eta)$ are then written as

$$X_m^{(1)}(\eta) = \Gamma_m^{(1)}(\theta_{11m}\widetilde{\mathrm{sn}}\ \alpha_m\eta + \theta_{12m}\widetilde{\mathrm{sn}}\ \beta_m\eta + \theta_{13m}\widetilde{\mathrm{sn}}\ \gamma_m\eta) \tag{19}$$

$$Y_m^{(1)}(\eta) = \Gamma_m^{(1)}(\theta_{21m}\widetilde{\mathrm{sn}}\ \alpha_m\eta + \theta_{22m}\widetilde{\mathrm{sn}}\ \beta_m\eta + \theta_{23m}\widetilde{\mathrm{sn}}\ \gamma_m\eta) \tag{20}$$

$$Z_m^{(1)}(\eta) = \Gamma_m^{(1)}(\theta_{31m}\widetilde{\mathrm{cs}}\ \alpha_m\eta + \theta_{32m}\widetilde{\mathrm{cs}}\ \beta_m\eta + \theta_{33m}\widetilde{\mathrm{cs}}\ \gamma_m\eta) \tag{21}$$

where $\Gamma_m^{(1)}$ and θ_{ijm} are unknown constants.

Case $\Delta_m > 0$

In this case, the characteristic roots become

$$\varepsilon_{m1,\,m2} = \pm\alpha_m \text{ or } \pm i\alpha_m, \qquad \varepsilon_{m3,\,m4} = \pm(\beta_m + i\gamma_m)$$
$$\varepsilon_{m5,\,m6} = \pm(\beta_m - i\gamma_m) \tag{22}$$

Analytical solutions may be written as

$$X_m^{(1)} = A_m \mathrm{cs}\ \alpha_m\eta + B_m \mathrm{sn}\ \alpha_m\eta + C_m \cosh\beta_m\eta\cos\gamma_m\eta + D_m \cosh\beta_m\eta\sin\gamma_m\eta + E_m \sinh\beta_m\eta\cos\gamma_m\eta + F_m \sinh\beta_m\eta\sin\gamma_m\eta \tag{23}$$

$$
\begin{aligned}
Y_m^{(1)} &= R_{m1}(A_m \mathrm{cs}\ \alpha_m\eta + B_m \mathrm{sn}\ \alpha_m\eta) \\
&+ (R^r_{m2}C_m + R^i_{m2}F_m)\cosh \beta_m\eta \cos \gamma_m\eta \\
&+ (R^r_{m2}D_m - R^i_{m2}E_m)\cosh \beta_m\eta \sin \gamma_m\eta \\
&+ (R^r_{m2}E_m + R^i_{m2}D_m)\sinh \beta_m\eta \cos \gamma_m\eta \\
&+ (R^r_{m2}F_m - R^i_{m2}C_m)\sinh \beta_m\eta \sin \gamma_m\eta
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
Z_m^{(1)} &= S_{m1}(B_m \mathrm{cs}\ \alpha_m\eta \pm A_m \mathrm{sn}\ \alpha_m\eta) \\
&+ (S^r_{m2}E_m + S^i_{m2}D_m)\cosh \beta_m\eta \cos \gamma_m\eta \\
&+ (S^r_{m2}F_m - S^i_{m2}C_m)\cosh \beta_m\eta \sin \gamma_m\eta \\
&+ (S^r_{m2}C_m + S^i_{m2}F_m)\sinh \beta_m\eta \cos \gamma_m\eta \\
&+ (S^r_{m2}D_m - S^i_{m2}E_m)\sinh \beta_m\eta \sin \gamma_m\eta
\end{aligned} \tag{25}
$$

where R^r_{m2} and R^i_{m2} are the real and imaginary parts of R_{m2}; S^r_{m2} and S^i_{m2} are the real and imaginary parts of S_{m2}, respectively; real constants R_{m1} and S_{m1} and complex constants R_{m2} and S_{m2} are known in terms of previously introduced constants and parameters.

Enforcing the boundary conditions along $\eta = 0$ and 1, one may determine all of the unknown constants in the preceding analytical solutions. The analytical solutions of the unknown functions are

$$
\begin{aligned}
X_m^{(1)}(\eta) &= \Gamma_m^{(1)}(\theta^*_{11m}\widetilde{\mathrm{sn}}\ \alpha_m\eta + \theta^*_{12m}\widetilde{\cosh}\ \beta_m\eta\ \widetilde{\sin}\ \gamma_m\eta \\
&+ \theta^*_{13m}\widetilde{\sinh}\ \beta_m\eta\ \widetilde{\cos}\ \gamma_m\eta)
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
Y_m^{(1)}(\eta) &= \Gamma_m^{(1)}(\theta^*_{21m}\widetilde{\mathrm{sn}}\ \alpha_m\eta + \theta^*_{22m}\widetilde{\cosh}\ \beta_m\eta\ \widetilde{\sin}\ \gamma_m\eta \\
&+ \theta^*_{23m}\widetilde{\sinh}\ \beta_m\eta\ \widetilde{\cos}\ \gamma_m\eta)
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
Z_m^{(1)}(\eta) &= \Gamma_m^{(1)}(\theta^*_{31m}\widetilde{\mathrm{cs}}\ \alpha_m\eta + \theta^*_{32m}\widetilde{\cosh}\ \beta_m\eta\ \widetilde{\cos}\ \gamma_m\eta \\
&+ \theta^*_{33m}\widetilde{\sinh}\ \beta_m\eta\ \widetilde{\sin}\ \gamma_m\eta)
\end{aligned} \tag{28}
$$

Summarizing both cases, the solution for the first building block may then be written as

$$
\begin{aligned}
W_1 &= \sum_{m\in\Omega_1} \Gamma_m^{(1)}(\theta_{11m}\widetilde{\mathrm{sn}}\ \alpha_m\eta + \theta_{12m}\widetilde{\mathrm{sn}}\ \beta_m\eta + \theta_{13m}\widetilde{\mathrm{sn}}\ \gamma_m\eta)\sin m\pi\xi \\
&+ \sum_{m\in\Omega_2} \Gamma_m^{(1)}(\theta^*_{11m}\widetilde{\mathrm{sn}}\ \alpha_m\eta + \theta^*_{12m}\widetilde{\cosh}\ \beta_m\eta\ \widetilde{\sin}\ \gamma_m\eta \\
&+ \theta^*_{13m}\widetilde{\sinh}\ \beta_m\eta\ \widetilde{\cos}\ \gamma_m\eta)\sin m\pi\xi
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\Psi_{\xi 1} &= \sum_{m\in\Omega_1} \Gamma_m^{(1)}(\theta_{21m}\widetilde{\mathrm{sn}}\ \alpha_m\eta + \theta_{22m}\widetilde{\mathrm{sn}}\ \beta_m\eta + \theta_{23m}\widetilde{\mathrm{sn}}\ \gamma_m\eta)\cos m\pi\xi \\
&+ \sum_{m\in\Omega_2} \Gamma_m^{(1)}(\theta^*_{21m}\widetilde{\mathrm{sn}}\ \alpha_m\eta + \theta^*_{22m}\widetilde{\cosh}\ \beta_m\eta\ \widetilde{\sin}\ \gamma_m\eta \\
&+ \theta^*_{23m}\widetilde{\sinh}\ \beta_m\eta\ \widetilde{\cos}\ \gamma_m\eta)\cos m\pi\xi
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
\Psi_{\eta 1} &= \sum_{m\in\Omega_1} \Gamma_m^{(1)}(\theta_{31m}\widetilde{\mathrm{cs}}\ \alpha_m\eta + \theta_{32m}\widetilde{\mathrm{cs}}\ \beta_m\eta + \theta_{33m}\widetilde{\mathrm{cs}}\ \gamma_m\eta)\sin m\pi\xi \\
&+ \sum_{m\in\Omega_2} \Gamma_m^{(1)}(\theta^*_{31m}\widetilde{\mathrm{cs}}\ \alpha_m\eta + \theta^*_{32m}\widetilde{\cosh}\ \beta_m\eta\ \widetilde{\cos}\ \gamma_m\eta \\
&+ \theta^*_{33m}\widetilde{\sinh}\ \beta_m\eta\ \widetilde{\cos}\ \gamma_m\eta)\sin m\pi\xi
\end{aligned} \tag{31}
$$

where Ω_1 is the domain in which $\Delta_m < 0$, and Ω_2 is the domain in which $\Delta_m > 0$.

Solution for the Second Building Block

Examining the Lévy type solutions for the first building block, one will notice that for the second building block the unknown function $Z_n^{(2)}$ is used in the expressions for $\Psi_{\xi 2}$, and $Y_n^{(2)}$ is used for $\Psi_{\eta 2}$. The reason for switching the unknown functions $Y_n^{(2)}$ and $Z_n^{(2)}$ in Lévy type solutions for the functionals $\Psi_{\xi 2}$ and $\Psi_{\eta 2}$ is that the differential equations possess a structure identical to that for the first building block, and no excessive work is needed for finding solutions of the unknown functions. Substituting the Lévy type solutions in Table 1 into the governing differential equations, one may obtain the following coefficients:

$$
\begin{pmatrix} a_{n1} \\ a_{n2} \\ a_{n3} \\ a_{n4} \end{pmatrix} =
\begin{bmatrix}
\dfrac{\kappa^2 a_{55}}{(\kappa^2 a_{55} - \phi^2 P_\xi / s^2)\phi} \\
\dfrac{(d_{12} + d_{66})\,\phi n\pi}{d_{66}} \\
-\dfrac{\kappa^2 s^2 a_{55}}{d_{11}} \\
-\dfrac{(d_{12} + d_{66})\,\phi n\pi}{d_{11}}
\end{bmatrix}
$$

$$
\begin{pmatrix} b_{n1} \\ b_{n2} \\ b_{n3} \\ b_{n4} \\ b_{n5} \end{pmatrix} =
\begin{bmatrix}
\dfrac{\lambda^4/s^2 + (P_\eta/s^2 - \kappa^2 a_{44})(\phi n\pi)^2}{\kappa^2 a_{55} - \phi^2 P_\xi/s^2} \\
-\dfrac{\kappa^2 a_{44}\phi n\pi}{\kappa^2 a_{55} - \phi^2 P_\xi/s^2} \\
-\dfrac{\kappa^2 s^2 a_{44}\phi n\pi}{d_{66}} \\
\dfrac{1}{d_{66}}\left[\dfrac{\lambda^4}{12s^2} - d_{22}(\phi n\pi)^2 - \kappa^2 s^2 a_{44}\right] \\
\dfrac{1}{d_{11}}\left[\dfrac{\lambda^4}{12s^2} - d_{66}(\phi n\pi)^2 - \kappa^2 s^2 a_{55}\right]
\end{bmatrix}
$$

The solution for the second building block is

$$
\begin{aligned}
W_2 &= \sum_{n\in\Omega_1} \Gamma_n^{(2)}(\theta_{11n}\widetilde{\mathrm{sn}}\ \alpha_n\xi + \theta_{12n}\widetilde{\mathrm{sn}}\ \beta_n\xi + \theta_{13n}\widetilde{\mathrm{sn}}\ \gamma_n\xi)\sin n\pi\eta \\
&+ \sum_{n\in\Omega_2} \Gamma_n^{(2)}(\theta^*_{11n}\widetilde{\mathrm{sn}}\ \alpha_n\xi + \theta^*_{12n}\widetilde{\cosh}\ \beta_n\xi\ \widetilde{\sin}\ \gamma_n\xi \\
&+ \theta^*_{13n}\widetilde{\sinh}\ \beta_n\xi\ \widetilde{\cos}\ \gamma_n\xi)\sin n\pi\eta
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
\Psi_{\xi 2} &= \sum_{n\in\Omega_1} \Gamma_n^{(2)}(\theta_{31n}\widetilde{\mathrm{cs}}\ \alpha_n\xi + \theta_{32n}\widetilde{\mathrm{cs}}\ \beta_n\xi + \theta_{33n}\widetilde{\mathrm{cs}}\ \gamma_n\xi)\sin n\pi\eta \\
&+ \sum_{n\in\Omega_2} \Gamma_n^{(2)}(\theta^*_{31n}\widetilde{\mathrm{cs}}\ \alpha_n\xi + \theta^*_{32n}\widetilde{\cosh}\ \beta_n\xi\ \widetilde{\cos}\ \gamma_n\xi \\
&+ \theta^*_{33n}\widetilde{\sinh}\ \beta_n\xi\ \widetilde{\sin}\ \gamma_n\xi)\sin n\pi\eta
\end{aligned} \tag{33}
$$

$$\Psi_{\eta 2} = \sum_{n \in \Omega_1} \Gamma_n^{(2)}(\theta_{21n}\widetilde{\text{sn}}\,\alpha_n\xi + \theta_{22n}\widetilde{\text{sn}}\,\beta_n\xi + \theta_{23n}\widetilde{\text{sn}}\,\gamma_n\xi)\cos n\pi\eta$$
$$+ \sum_{n \in \Omega_2} \Gamma_n^{(2)}(\theta^*_{21n}\widetilde{\text{sn}}\,\alpha_n\xi + \theta^*_{22n}\widetilde{\cosh}\,\beta_n\xi\,\widetilde{\sin}\,\gamma_n\xi$$
$$+ \theta^*_{23n}\widetilde{\sinh}\,\beta_n\xi\,\widetilde{\cos}\,\gamma_n\xi)\cos n\pi\eta \qquad (34)$$

where Ω_1 is the domain for $\Delta_n < 0$, and Ω_2 is the domain for $\Delta_n > 0$.

Solutions for the Third and Fourth Building Block

Solutions for the third and fourth building blocks may be obtained following the same procedure used for the first and second building blocks. However, the solution procedure for these two building blocks may be greatly simplified if the similarity of the imposed boundary conditions between the first and third building blocks, and between the second and fourth building blocks, is taken into consideration. The solution for the third building block may be inferred from the solution for the first building block by replacing η with η^* (= $1 - \eta$). The solution for the fourth building block may be obtained from the solution for the second building block by replacing ξ with ξ^* (= $1 - \xi$). The solution for the third building block is

$$W_3 = \sum_{m \in \Omega_1} \Gamma_m^{(3)}(\theta_{11m}\widetilde{\text{sn}}\,\alpha_m\eta^* + \theta_{12m}\widetilde{\text{sn}}\,\beta_m\eta^* + \theta_{13m}\widetilde{\text{sn}}\,\gamma_m\eta^*)\sin m\pi\xi$$
$$+ \sum_{m \in \Omega_2} \Gamma_m^{(3)}(\theta^*_{11m}\widetilde{\text{sn}}\,\alpha_m\eta^* + \theta^*_{12m}\widetilde{\cosh}\,\beta_m\eta^*\,\widetilde{\sin}\,\gamma_m\eta^*$$
$$+ \theta^*_{13m}\widetilde{\sinh}\,\beta_m\eta^*\,\widetilde{\cos}\,\gamma_m\eta^*)\sin m\pi\xi \qquad (35)$$

$$\Psi_{\xi 3} = \sum_{m \in \Omega_1} \Gamma_m^{(3)}(\theta_{21m}\widetilde{\text{sn}}\,\alpha_m\eta^* + \theta_{22m}\widetilde{\text{sn}}\,\beta_m\eta^* + \theta_{23m}\widetilde{\text{sn}}\,\gamma_m\eta^*)\cos m\pi\xi$$
$$+ \sum_{m \in \Omega_2} \Gamma_m^{(3)}(\theta^*_{21m}\widetilde{\text{sn}}\,\alpha_m\eta^* + \theta^*_{22m}\widetilde{\cosh}\,\beta_m\eta^*\,\widetilde{\sin}\,\gamma_m\eta^*$$
$$+ \theta^*_{23m}\widetilde{\sinh}\,\beta_m\eta^*\,\widetilde{\cos}\,\gamma_m\eta^*)\cos m\pi\xi \qquad (36)$$

$$\Psi_{\eta 3} = -\sum_{m \in \Omega_1} \Gamma_m^{(3)}(\theta_{31m}\widetilde{\text{cs}}\,\alpha_m\eta^* + \theta_{32m}\widetilde{\text{cs}}\,\beta_m\eta^* + \theta_{33m}\widetilde{\text{cs}}\,\gamma_m\eta^*)\sin m\pi\xi$$
$$- \sum_{m \in \Omega_2} \Gamma_m^{(3)}(\theta^*_{31m}\widetilde{\text{cs}}\,\alpha_m\eta^* + \theta^*_{32m}\widetilde{\cosh}\,\beta_m\eta^*\,\widetilde{\cos}\,\gamma_m\eta^*$$
$$+ \theta^*_{33m}\widetilde{\sinh}\,\beta_m\eta^*\,\widetilde{\sin}\,\gamma_m\eta^*)\sin m\pi\xi \qquad (37)$$

The solution for the fourth building block is

$$W_4 = \sum_{n \in \Omega_1} \Gamma_n^{(4)}(\theta_{11n}\widetilde{\text{sn}}\,\alpha_n\xi^* + \theta_{12n}\widetilde{\text{sn}}\,\beta_n\xi^* + \theta_{13n}\widetilde{\text{sn}}\,\gamma_n\xi^*)\sin n\pi\eta$$
$$+ \sum_{n \in \Omega_2} \Gamma_n^{(4)}(\theta^*_{11n}\widetilde{\text{sn}}\,\alpha_n\xi^* + \theta^*_{12n}\widetilde{\cosh}\,\beta_n\xi^*\,\widetilde{\sin}\,\gamma_n\xi^*$$
$$+ \theta^*_{13n}\widetilde{\sinh}\,\beta_n\xi^*\,\widetilde{\cos}\,\gamma_n\xi^*)\sin n\pi\eta \qquad (38)$$

$$\Psi_{\xi 4} = -\sum_{n \in \Omega_1} \Gamma_n^{(4)}(\theta_{31n}\widetilde{\text{cs}}\,\alpha_n\xi^* + \theta_{32n}\widetilde{\text{cs}}\,\beta_n\xi^* + \theta_{33n}\widetilde{\text{cs}}\,\gamma_n\xi^*)\sin n\pi\eta$$
$$- \sum_{n \in \Omega_2} \Gamma_n^{(4)}(\theta^*_{31n}\widetilde{\text{cs}}\,\alpha_n\xi^* + \theta^*_{32n}\widetilde{\cosh}\,\beta_n\xi^*\,\widetilde{\cos}\,\gamma_n\xi^*$$
$$+ \theta^*_{33n}\widetilde{\sinh}\,\beta_n\xi^*\,\widetilde{\sin}\,\gamma_n\xi^*)\sin n\pi\eta \qquad (39)$$

$$\Psi_{\eta 4} = \sum_{n \in \Omega_1} \Gamma_n^{(4)}(\theta_{21n}\widetilde{\text{sn}}\,\alpha_n\xi^* + \theta_{22n}\widetilde{\text{sn}}\,\beta_n\xi^* + \theta_{23n}\widetilde{\text{sn}}\,\gamma_n\xi^*)\cos n\pi\eta$$
$$+ \sum_{n \in \Omega_2} \Gamma_n^{(4)}(\theta^*_{21n}\widetilde{\text{sn}}\,\alpha_n\xi^* + \theta^*_{22n}\widetilde{\cosh}\,\beta_n\xi^*\,\widetilde{\sin}\,\gamma_n\xi^*$$
$$+ \theta^*_{23n}\widetilde{\sinh}\,\beta_n\xi^*\,\widetilde{\cos}\,\gamma_n\xi^*)\cos n\pi\eta \qquad (40)$$

Formulation of Eigenvalue Matrices

To obtain a solution for a clamped plate, the solutions for all four building blocks are superimposed. The superimposed solution is written as

$$W(\xi,\eta) = \sum_{i=1}^{4} W_i(\xi,\eta), \qquad \Psi_\xi(\xi,\eta) = \sum_{i=1}^{4} \Psi_{\xi i}(\xi,\eta)$$
$$\Psi_\eta(\xi,\eta) = \sum_{i=1}^{4} \Psi_{\eta i}(\xi,\eta) \qquad (41)$$

It is obvious that the preceding superimposed solution containing four sets of unknown constants $\Gamma_m^{(1)}$, $\Gamma_n^{(2)}$, $\Gamma_m^{(3)}$, and $\Gamma_n^{(4)}$ also satisfies the governing differential equations and two of the three boundary conditions prescribed on any of the four clamped edges. For the superimposed solution to be a solution for the clamped laminate, the following boundary conditions released during the construction of building blocks must be enforced:

$$\Psi_\eta(\xi,1) = \Psi_\xi(1,\eta) = \Psi_\eta(\xi,0) = \Psi_\xi(0,\eta) = 0 \qquad (42)$$

Since the solution for each building block contains k unknown constants present in the driving quantity expressions, enforcing the released boundary conditions for the superimposed solution yields the following $4k$ homogeneous algebraic equations:

$$\begin{bmatrix} \boldsymbol{A}_{11}(m,m) & \boldsymbol{A}_{12}(m,n) & \boldsymbol{A}_{13}(m,m) & \boldsymbol{A}_{14}(m,n) \\ \boldsymbol{A}_{21}(n,m) & \boldsymbol{A}_{22}(n,n) & \boldsymbol{A}_{23}(n,m) & \boldsymbol{A}_{24}(n,n) \\ \boldsymbol{A}_{31}(m,m) & \boldsymbol{A}_{32}(m,n) & \boldsymbol{A}_{33}(m,m) & \boldsymbol{A}_{34}(m,n) \\ \boldsymbol{A}_{41}(n,m) & \boldsymbol{A}_{42}(n,n) & \boldsymbol{A}_{43}(n,m) & \boldsymbol{A}_{44}(n,n) \end{bmatrix} \begin{Bmatrix} \Gamma^{(1)} \\ \Gamma^{(2)} \\ \Gamma^{(3)} \\ \Gamma^{(4)} \end{Bmatrix} = \boldsymbol{0} \qquad (43)$$

where $\boldsymbol{A}_{ij}$ are submatrices of dimensions $k \times k$, and $\Gamma^{(i)}$ are column arrays containing the k unknown coefficients in the expressions for the driving bending moment of the ith building block. The four sets of column arrays constitute a modal vector for vibration and buckling analysis. Each of the preceding submatrices has certain physical meaning; $\boldsymbol{A}_{ij}$ represents the contribution of the jth building block to the released boundary condition on the ith edge. For a rectangular plate, edges $\eta = 1$, $\xi = 1$, $\eta = 0$, and $\xi = 0$ are numbered 1, 2, 3, and 4, respectively. Among the 16 submatrices, 8 are diagonal, and the remaining 8 are full. All of the submatrices may be determined from Eq. (42).

The preceding equations are the general algebraic equations for a fully clamped laminate. Solutions for laminates involving simply supported edges may be obtained from the solution for the clamped laminate by eliminating the building blocks associated with the simply supported edges. For example, to obtain the algebraic equations for a laminate with edge $\xi = 0$ simply supported and the remaining edges clamped, one deletes from the general algebraic equations all seven submatrices and one column array associated with the fourth building block and the zero slope boundary conditions imposed on the fourth edge. These submatrices are $\boldsymbol{A}_{14}(m,n)$, $\boldsymbol{A}_{24}(n,n)$, $\boldsymbol{A}_{34}(m,n)$, $\boldsymbol{A}_{44}(n,n)$, $\boldsymbol{A}_{41}(m,n)$, $\boldsymbol{A}_{42}(n,n)$, and $\boldsymbol{A}_{43}(n,m)$. The dimensions of the restructured algebraic equations are consequently reduced to $3k \times 3k$.

The analysis of free vibration and buckling is now reduced to seek zero roots of the determinant of the coefficient matrix or eigenvalue matrix for frequency parameter or critical buckling load and to solve the algebraic equations for mode shapes of vibrating or buckled laminates.

A laminate central axis is the axis of symmetry if the two edges parallel to this axis have identical boundary conditions. Mode shapes of the laminate are therefore either symmetric or antisymmetric with respect to the axis of symmetry. To facilitate the symmetric or antisymmetric analysis of mode shapes, the general eigenvalue matrices must be restructured according to the mathematical conditions derived from the symmetry or antisymmetry of mode shapes. For example, if a mode shape is symmetric with respect to the ξ_c axis, then $W(\xi, \eta) = W(\xi, \eta^*)$; if it is antisymmetric with respect to ξ_c, then $W(\xi, \eta) = -W(\xi, \eta^*)$. To insure these conditions, the two summation variables and elements of the modal vector must be subjected to certain constraints given in Table 3.

Assuming that k terms are used in all Lévy solutions, the dimensions of the ensuing algebraic equations are reduced due to symmetry and antisymmetry of mode shapes. However, this reduction is at the cost of restructuring Eq. (43). As an illustrative example, let us look at a fully clamped plate. Since the boundary conditions along $\eta = 1$ and 0 are identical, and the boundary conditions along $\xi = 1$ and 0 are also identical, mode shapes for the fully clamped plate must be 1) symmetric about both ξ_c and η_c, 2) symmetric about ξ_c and antisymmetric about η_c, 3) antisymmetric about ξ_c and symmetric about η_c, and 4) antisymmetric about both ξ_c and η_c.

To simplify the mathematical representations of the restructured algebraic equations, the upper limit k in the truncated Lévy type solutions is taken to be an even number. Applying the conditions in Table 3, one obtains the following restructured algebraic equations for the four families of mode shapes:

$$\begin{bmatrix} A_{11}(m_o, m_o) + A_{13}(m_o, m_o) & A_{12}(m_o, n_o) + A_{14}(m_o, n_o) \\ A_{21}(n_o, m_o) + A_{23}(n_o, m_o) & A_{22}(n_o, n_o) + A_{24}(n_o, n_o) \end{bmatrix} \begin{Bmatrix} \overset{o}{\Gamma}{}^{(1)} \\ \overset{o}{\Gamma}{}^{(2)} \end{Bmatrix} = \mathbf{0} \tag{44}$$

$$\begin{bmatrix} A_{11}(m_e, m_e) + A_{13}(m_e, m_e) & A_{12}(m_e, n_o) - A_{14}(m_e, n_o) \\ A_{21}(n_o, m_e) + A_{23}(n_o, m_e) & A_{22}(n_o, n_o) - A_{24}(n_o, n_o) \end{bmatrix} \begin{Bmatrix} \overset{o}{\Gamma}{}^{(1)} \\ \overset{o}{\Gamma}{}^{(2)} \end{Bmatrix} = \mathbf{0} \tag{45}$$

$$\begin{bmatrix} A_{11}(m_o, m_o) - A_{13}(m_o, m_o) & A_{12}(m_o, n_e) + A_{14}(m_o, n_e) \\ A_{21}(n_e, m_o) - A_{23}(n_e, m_o) & A_{22}(n_e, n_e) + A_{23}(n_e, n_e) \end{bmatrix} \begin{Bmatrix} \overset{o}{\Gamma}{}^{(1)} \\ \overset{e}{\Gamma}{}^{(2)} \end{Bmatrix} = \mathbf{0} \tag{46}$$

$$\begin{bmatrix} A_{11}(m_e, m_e) - A_{13}(m_e, m_e) & A_{12}(m_e, n_e) - A_{14}(m_e, n_e) \\ A_{21}(n_e, m_e) - A_{23}(n_e, m_e) & A_{22}(n_e, n_e) - A_{23}(n_e, n_e) \end{bmatrix} \begin{Bmatrix} \overset{e}{\Gamma}{}^{(1)} \\ \overset{e}{\Gamma}{}^{(2)} \end{Bmatrix} = \mathbf{0} \tag{47}$$

where $m_o, n_o = 1, 3, 5, \ldots, k-1$; $m_e, n_e = 2, 4, 6, \ldots, k$; $\overset{e}{\Gamma}{}^{(i)} = \Gamma^{(i)}(m = m_e$ or $n = n_e)$; and $\overset{o}{\Gamma}{}^{(i)} = \Gamma^{(i)}(m = m_o$ or $n = n_o)$.

Table 3 Conditions for symmetric and antisymmetric modes

Modes shapes	Conditions	
Symmetric about ξ_c axis	$n = 1, 3, 5, \ldots;$	$\Gamma_m^{(3)} = \Gamma_m^{(1)}$
Symmetric about η_c axis	$m = 1, 3, 5, \ldots;$	$\Gamma_n^{(4)} = \Gamma_n^{(2)}$
Antisymmetric about ξ_c axis	$n = 2, 4, 6, \ldots;$	$\Gamma_m^{(3)} = -\Gamma_m^{(1)}$
Antisymmetric about η_c axis	$m = 2, 4, 6, \ldots;$	$\Gamma_n^{(4)} = -\Gamma_n^{(2)}$

Table 4 Convergency test for eigenvalues of a clamped square plate

	Number of terms used k						
Mode	2	3	4	5	6	7	8
1	21.387	21.447	21.447	21.448	21.448	21.448	21.448
2	29.942	29.946	29.979	29.979	29.980	29.980	29.980
3	40.366	40.388	40.388	40.388	40.389	40.389	40.389
4	42.890	44.483	44.483	44.495	44.495	44.495	44.495
5	45.765	45.765	45.783	45.783	45.784	45.784	45.784
6	55.568	56.595	56.596	56.604	56.604	56.604	56.604

Numerical Results and Discussions

It is unthinkable to consider all possible values of the basic parameters appearing in free vibration and buckling analysis. These include fabrication parameters $\mathfrak{M}$ and $\mathfrak{N}$, defined as

$$\mathfrak{M} = \frac{\Sigma_{p=\text{even}}(h_p - h_{p-1})}{\Sigma_{p=\text{odd}}(h_p - h_{p-1})}, \qquad \mathfrak{N} = \frac{\Sigma_{p=\text{even}}(h_p^3 - h_{p-1}^3)}{\Sigma_{p=\text{odd}}(h_p^3 - h_{p-1}^3)}$$

geometry parameters ϕ and s, and material parameters E_1/E_2, ν_{12}, G_{12}/E_2, G_{23}/E_2, and G_{31}/E_2. In this paper, numerical results are obtained for the case of a rectangular laminate (0/90/0 deg). All laminas are assumed to be of the same thickness and density and to be made of the same orthotropic material. Except for the purpose of comparison, all eigenvalues are calculated for parameters of the following values: 1) fabrication parameters: $\mathfrak{M} = 0.5$ and $\mathfrak{N} = 0.0385$; 2) geometry parameters: $s = 10$, $\phi = 0.5, 0.75, 1.0, 1.5$, and 2.0; 3) material (typical high-modulus graphite/epoxy) property parameters: $\nu_{12} = 0.25$, $E_1/E_2 = 40$, $G_{12}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, and $G_{31}/E_2 = 0.6$; and 4) load ratio: $\alpha = P_y/P_x = 1.0$ (buckling analysis only). The nondimensional rigidity parameters may be expressed in terms of material properties and fabrication parameters. They are written as

$$a_{44} = \frac{G_{23}/E_2 + (G_{31}/E_2)\mathfrak{M}}{1 + \mathfrak{M}} \qquad a_{55} = \frac{(G_{23}/E_2)\mathfrak{M} + G_{31}/E_2}{1 + \mathfrak{M}}$$

$$d_{11} = \frac{E_1/E_2 + \mathfrak{N}}{12(1 - \nu_{12}\nu_{21})(1 + \mathfrak{N})} \qquad d_{12} = \frac{\nu_{12}}{12(1 - \nu_{12}\nu_{21})}$$

$$d_{22} = \frac{1 + N(E_1/E_2)}{12(1 - \nu_{12}\nu_{21})(1 + \mathfrak{N})} \qquad d_{66} = \frac{G_{12}}{12E_2}$$

In running the computer code for the free vibration and buckling analysis, one finds that only a few terms in the Lévy type solutions are needed to achieve five-digit accuracy. Table 4 gives the frequency parameters of a clamped square laminated plate for different values of k. Reviewing the first six eigenvalues, one may conclude that six terms in the Lévy type solutions will guarantee five-digit accuracy for the computed eigenvalues. It was decided to employ $k = 8$ throughout this paper.

To understand how well the IPT works for laminated plates, results of IPT are compared with those obtained from other plate theories. Tables 5 and 6 give a comparison of fundamental frequency parameter for a square plate of different values of s and E_1/E_2. In the two tables, notations SSSC and SCSC are used, where C represents clamp and S simple support. The four-character string denotes the boundary conditions along the edges $\eta = 1$, $\xi = 1$, $\eta = 0$, and $\xi = 0$, respectively. From these two tables, one can conclude that 1) the CPT gives inappropriate eigenvalues, especially for plates with high in-plane moduli and low length-to-thickness ratios; 2) the IPT gives very good results for different in-plane moduli and length-to-thickness ratios, and the error is within 2% even when the length-to-thickness ratio is as low as 5; and 3) eigenvalues increase with the increase of ratios E_1/E_2 and s.

Eigenvalues of the first six vibration modes are calculated for plates with five different boundary conditions. They are presented in Tables 7–11. As can be predicted, eigenvalues of plates with clamped edges are higher than those plates whose corresponding clamped edges become simply supported. For a clamped square plate with $E_1/E_2 = 40$ and $s = 10$, the fundamental eigenvalue is 12% higher than that for the same plate with edge $\xi = 0$ simply supported. In these tables, the letter "s" or "a" within parentheses beside an eigenvalue represents the symmetry and antisymmetry of the vibration mode corresponding to that eigenvalue. In case there are two symmetric (antisymmetric) axes, the first letter s or a means symmetry or antisymmetry about ξ_c, and the second letter s or a preceded by a hyphen means symmetry or antisymmetry about η_c. For a plate having only one symmetric axis, a single s or a means symmetry or antisymmetry about this axis.

Table 12 gives a comparison of critical buckling loads for a laminated square plate with different length-to-thickness ratios. The tabulated numerical results again indicate that the CPT fails to provide reliable critical buckling loads, particularly for lower s. For example, for an SSSC square plate with $s = 5$, the critical buckling load is 23.381 by CPT compared with 5.888 by the higher order shear deformation theory (HSDT) and 6.004 by the IPT.

Table 5 Comparison of eigenvalues for a square plate ($s = 10$)

Theories used	E_1/E_2 (SSSC) 20	30	40	E_1/E_2 (SCSC) 20	30	40
HSDPT[4]	15.036	16.458	17.427	18.124	19.448	20.315
HSPT[4]	14.888	16.203	17.069	17.727	18.830	19.500
CPT[4]	20.610	24.870	28.501	19.166	35.431	40.743
IPT	15.059	16.430	17.336	18.017	19.187	19.898

Table 6 Comparison eigenvalues for a square plate ($E_1/E_2 = 40$)

Theories used	s (SSSC) 5	10	15	s (SCSC) 5	10	15
HSDPT[4]	11.156	17.427	21.325	12.333	20.315	26.183
HSPT[4]	10.576	17.069	21.131	11.164	19.500	25.652
CPT[4]	28.501	28.501	28.501	40.743	40.743	40.743
IPT	10.768	17.336	21.373	11.411	19.898	26.072

Table 7 First six eigenvalues λ^2 of CCCC plates

Mode	Plate aspect ratio ϕ 0.50	0.75	1.00	1.50	2.00
1	19.48(s-s)	20.09(s-s)	21.45(s-s)	26.87(s-s)	35.18(s-s)
2	20.50(a-s)	23.92(a-s)	29.98(a-s)	43.61(s-a)	49.23(s-a)
3	23.12(s-s)	31.91(s-s)	40.39(s-a)	47.11(a-s)	67.03(a-s)
4	27.69(a-s)	39.63(s-a)	44.50(s-s)	58.64(a-a)	68.57(s-s)
5	34.02(s-s)	41.92(a-a)	45.78(a-a)	64.63(s-s)	75.70(a-a)
6	39.27(s-a)	43.13(a-s)	50.60(s-a)	74.40(s-s)	89.41(s-a)

Table 8 First six eigenvalues λ^2 of CCCS plates

Mode	Plate aspect ratio ϕ 0.50	0.75	1.00	1.50	2.00
1	16.77(s)	17.51(s)	19.09(s)	25.09(s)	33.88(s)
2	18.03(a)	21.94(a)	28.50(a)	42.90(s)	48.61(s)
3	21.06(s)	30.57(s)	39.61(s)	46.27(a)	66.49(a)
4	26.09(a)	38.82(s)	43.62(s)	58.17(a)	68.21(s)
5	32.82(s)	41.19(a)	45.14(a)	64.24(s)	75.36(a)
6	38.45(s)	42.23(a)	56.12(s)	73.95(s)	89.27(s)

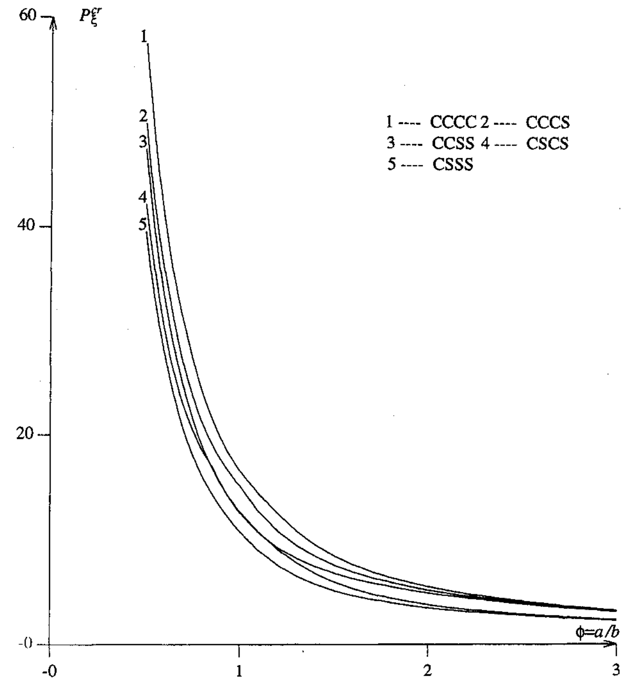

Fig. 2 Critical buckling load of rectangular laminates vs ϕ ($\alpha = 1.0$).

Table 9 First six eigenvalues λ^2 of CCSS plates

Mode	Plate aspect ratio ϕ 0.50	0.75	1.00	1.50	2.00
1	16.67	17.10	18.03	21.94	28.50
2	17.64	20.65	26.09	41.19	45.14
3	20.26	28.63	39.12	42.23	61.86
4	24.94	38.64	40.82	55.07	65.84
5	31.46	40.11	43.69	63.14	71.38
6	38.40	40.53	54.02	73.12	87.51

Table 10 First six eigenvalues λ^2 of CSCS plates

Mode	Plate aspect ratio ϕ 0.50	0.75	1.00	1.50	2.00
1	14.11(s-s)	15.02(s-s)	16.87(s-s)	23.48(s-s)	32.73(s-s)
2	15.66(a-s)	20.12(a-s)	27.18(a-s)	42.02(s-a)	47.86(s-a)
3	19.16(s-s)	29.38(s-s)	38.63(s-a)	45.53(a-s)	66.02(a-s)
4	24.66(a-s)	37.81(s-a)	42.85(s-s)	57.60(a-a)	67.88(s-s)
5	31.75(s-s)	40.29(a-a)	44.35(a-a)	63.88(s-s)	74.95(a-a)
6	37.42(s-a)	41.44(a-s)	55.55(s-a)	73.55(s-s)	89.11(s-a)

Table 11 First six eigenvalues λ^2 of CSSS plates

Mode	Plate aspect ratio ϕ 0.50	0.75	1.00	1.50	2.00
1	13.99(s)	14.54(s)	15.66(s)	20.12(s)	27.18(s)
2	15.21(s)	18.71(s)	24.66(s)	40.29(a)	44.35(a)
3	18.28(s)	27.37(s)	38.13(a)	41.44(s)	61.38(s)
4	23.44(s)	37.63(a)	40.01(s)	54.49(a)	65.50(s)
5	30.35(s)	39.29(a)	42.88(a)	62.78(s)	70.97(a)
6	37.95(a)	39.61(a)	53.43(a)	70.15(s)	85.92(s)

Table 12 Comparison of critical buckling loads for a square plate

Plate theories	s (SSSC) 5	10	15	s (SCSC) 5	10	15
HSDPT[4]	5.781	11.534	15.375	6.219	13.483	19.706
HSPT[4]	5.888	11.634	15.447	6.163	13.288	19.526
CPT[4]	23.381	23.381	23.381	34.45	34.454	34.454
IPT	6.004	11.753	15.594	6.232	13.509	19.862

The relationship between critical buckling load and plate aspect ratio is depicted in Fig. 2. From this figure, one can see that the critical buckling loads for all five different combinations of boundary conditions tend to decrease with the increase of the plate aspect ratio. Like plates made of isotropic materials, laminated plates may also buckle in different modes. Buckling modes depend not only on the plate aspect ratio but also on the load ratio, material properties, and boundary conditions.

Conclusions

This paper proposed uses of the method of superposition to analyze free vibration and buckling of symmetric cross-ply laminated rectangular plates. The analytical solutions given by the method satisfy the governing differential equations exactly and the boundary conditions to an arbitrary degree of accuracy. The method may be applied to laminated plates with various other boundary conditions and interior supports.

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